The Renormalization-Group method

A solid or liquid material may undergo a dramatic change in its properties depending on critical values of external parameters like temperature or pressure. A magnet for example loses its permanent magnetization at the so-called Curie-point. The materials behavior near such a **critical point** can be quantitatively described by a mathematical technique, called the **Renormalization-Group** method. It was originally introducing in quantum-field-theory, and later adapted to problems of statistical physics and condensed matter.

Here we demonstrate this technique by a simple example, the **percolation-problem**.

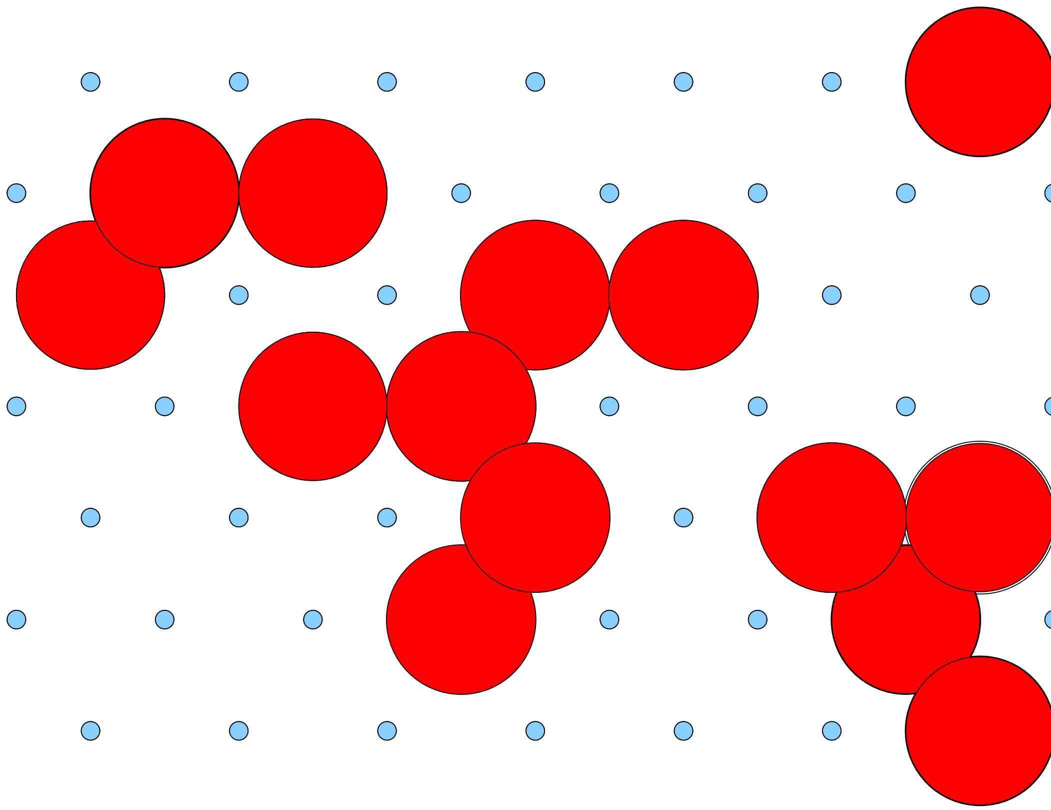
Hot water poured into a coffee-filter filled with ground coffee percolates through the coffee-powder and the filter until it drips out into the cup. Another example is electric current "percolating" through a random network.

Let us assume a regular grid of insulating points. If we now put circular copper disks like coins onto those grid-points, which are slightly larger than the distance between the grid-points, then electricity can flow through these copper-discs from one side of the grid to the other. Obviously, if only a very small number of grid-points at random carry a copper-disc, electricity cannot flow through the system. This is only possible from a certain **critical coverage** on, the percolation point. Increasing the probability of coverage from very small values to its critical value, two-dimensional clusters of electrically conducting grid-point will appear at increasing size, until at the percolation point one cluster of infinite size exists, allowing the current to go through the whole system. The diameter of a typical cluster is called the correlation length, which goes to infinity when the percolation point is reached.

The renormalization-group idea is, to replace a group of neighboring conducting sites by a single conducting site, and then rescale the lengths of the system accordingly. So if one replaces N neighboring conducting sites by just one site, then in 2 dimensions the lengths have to be changed by a factor $N^{1/2}$ which is square-root of N. Imagine to sit inside a helicopter and look down onto the grid. The renormalization-process corresponds to increasing the height of the helicopter above ground. At low coverage of the grid with conducting discs the visible clusters become smaller and smaller as the helicopter climbs, and ultimately it appears like a non-conducting grid. But at the critical percolation point, where the cluster-size has become infinite, the picture from above always looks approximately the same no matter how high the helicopter has climbed.

This process is illustrated by the following images which also gives a **renormalization-group**-calculation for the dependence of the **correlation-length** ξ upon the probability of coverage **P**. (The calculation here is "very good" but not "rigorous", since the definition of "percolation" for three neighboring points is very plausible but not rigorous, which would require to take much larger clusters into account).

Percolation

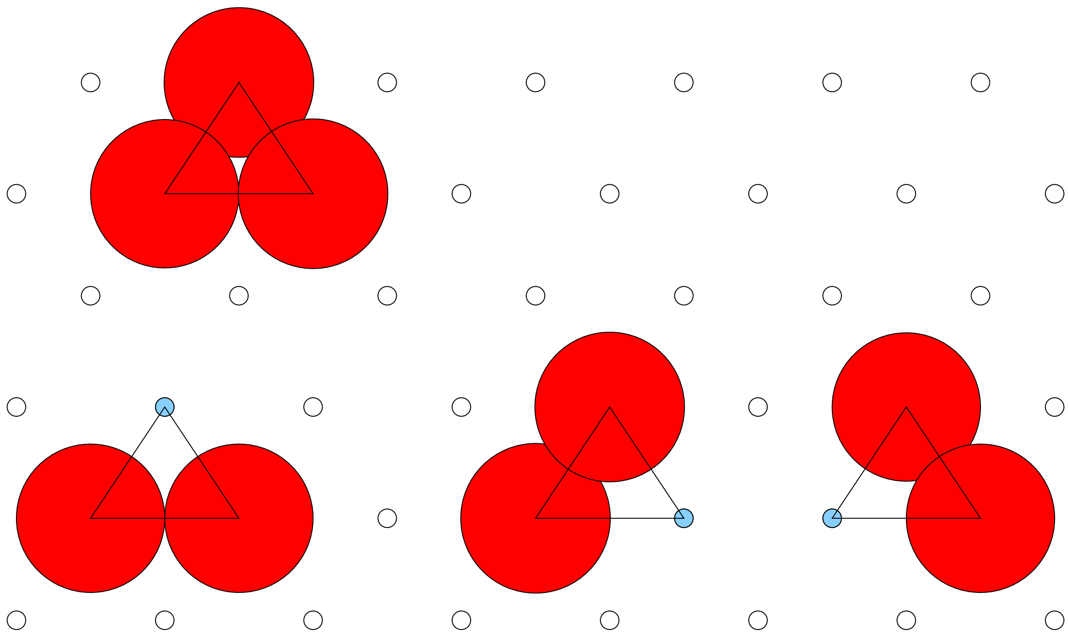


Percolation on Triangular Lattice

Probability P that a blue grid-point is occupied by a red circle here is $P = 14/52 = 0.269$

Renormalization of Percolation

Percolation-Probability for one Triangle



$$\tilde{P} = P^3 + 3\,P^2\,(1 - P)$$

$\tilde{P}$ is the new effective probability after reducing the number of grid-points by three to one.

Renormalization Group

Example: Percolation

$P =$ probability, that one point is percolating
$\tilde{P} =$ probability, that a triangle is percolating
$\xi =$ diameter of typical percolation-cluster

Renormalize one triangle to one point:

$$\tilde{P} = P^3 + 3P^2(1 - P)$$

The compression of 3 points in the plane to 1 point is a compression of the area by a factor 3. This is a change in length scale by $b = \sqrt{3}$.

Renormalization-group transformation:

$$\tilde{P} = R(P); \qquad \tilde{\xi} = \xi \ / \ b$$

Fixed-point: $\tilde{P} = P = P_c = 1/2$ (and at 0,1).

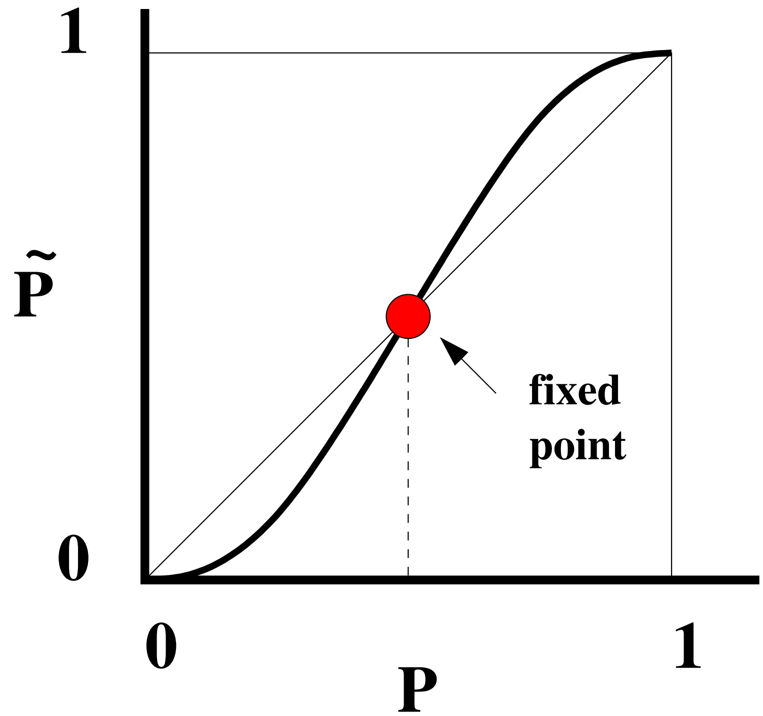
The correlation-length ξ goes to infinity as $P \to P_c$.
Critical exponent ν (definition): $\xi \approx 1/|P - P_c|^\nu$
This gives $\xi/\tilde{\xi} = |(\tilde{P} - P_c)/(P - P_c)|^\nu$
and in numbers: $\sqrt{3} = (3/2)^\nu$, since near P_c we have the linearized renormalization transform:

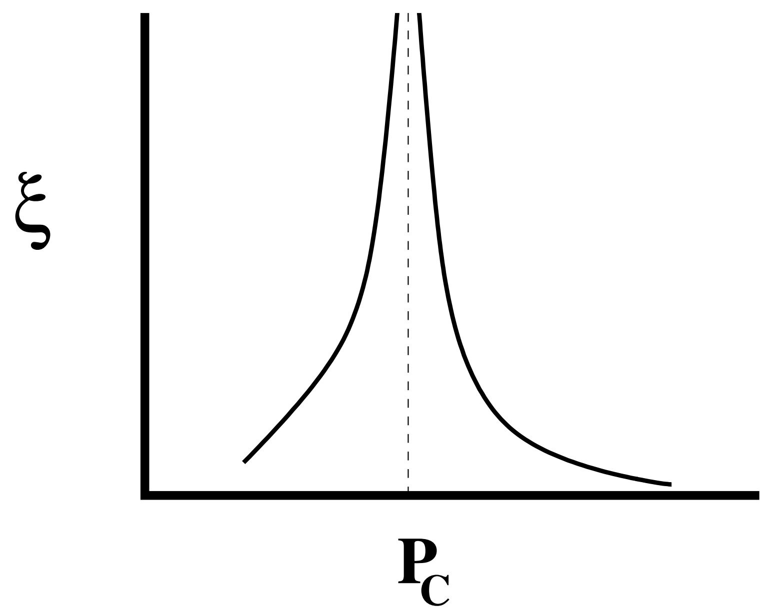
$$\tilde{P} = R(P) \approx P_c + (3/2)(P - P_c).$$

The critical exponent ν therefore is:

$$\nu = log(\sqrt{3})/log(3/2) = 1.3547$$

Renormalization and Correlation Length





$\xi \sim 1/|P - P_c|^\nu$ with $\nu \approx 1.354$ is the correlation length ξ which goes to infinity at the percolation-threshold $P_c = 1/2$.