Hydrodynamics - Overview

Navier-Stokes Equations

The flow of water or air seen from an observer in a fixed coordinate system is described by the Navier-Stokes equations [1,2,3,4,5]

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\ \nabla)\ \vec{u}\ = -\frac{1}{\rho}\nabla p + \nu \Delta \vec{u} + \vec{f}(\vec{x}) \tag{1}$$

These are the equations for the three vector-components of the velocity field $\vec{u}$ depending upon space $\vec{x}$ and time t. p is the pressure, ρ is the density of the fluid, $\vec{f}$ is a force density, for example the gravitational force. Specifically these equations describe momentum-conservation.
Assuming that the fluid is almost incompressible we require the divergence of the velocity field to vanish

$$\nabla \vec{u} = 0 \tag{2}$$

This equation-system looks like a differential equation, but in practice it has to be treated like an integral equation: the pressure field depends on the velocity field and *vice versa*. This can be seen by applying the divergence operator (2) upon the whole equation (1), which gives a Poisson-equation for the pressure. This equation for the pressure must be fulfilled at any instant of time. But the eigenfunctions of the resulting Laplace-operator are long-ranged and so every point of the fluid volume couples instantaneously (more precisely: with sound velocity) to every other point.

Turbulence

Kolmogorov K41-Theory

The Navier-Stokes equation(s) can be made dimensionless by introducing a characteristic velocity v and a characteristic length scale L. One then can reformulate the equation so that there is only on dimensionless parameter left, the *Reynolds number*:

$$Re = \frac{Lv}{\nu} \tag{3}$$

with ν being the kinematic viscosity. (air: $\nu_a \approx 0.16cm^2/s$, water: $\nu_w \approx 0.01cm^2/s$). Flows at the same Reynolds numbers are similar. When the Reynolds number exceeds values of about $Re \approx 1000$ the flow becomes turbulent.

In a fully turbulent flow the fluid flow consists of eddys of all sizes. According to Kolomogorovs theory of 1941 (K41) the energy is transported from large to small eddys, where the coupling between the eddies is most efficient only between eddies of about the same order of scale. Denoting $k = 2\pi/\lambda$ as the wavenumber for wavelength or eddy-size λ, the energy density $E(k)$ in each wave-length interval dk gives integrated the total energy content of the turbulent volume

$$E = \int E(k)\ dk \tag{4}$$

When the total energy flow is ϵ, we have from dimensional analysis directly

$$E(k) \sim \epsilon^{2/3}\ k^{-5/3} \tag{5}$$

This is the famous $E(k) \sim k^{-5/3}$ result of the K41-theory for the energy density per k-mode.
The result is obtained from dimensional analysis as follows. The dimensions of energy (per mass unit) are $[E] = length^2/time^2$. The dimensions of energy flow are $[\epsilon] = energy/(mass\ time)$, and the dimension of wavenumber is $[k] = 1/length$. Making the Ansatz $E(k) \sim \epsilon^x\ k^y$ and inserting the appropriate dimensions into eq.(4), one arrives at $[E(k)] = length^3/time^2$. The exponents of *length* and *time* separately must be chosen such, that eq.(4) is fulfilled. This gives immediately $2 = 3x$ and $3 = 2x - y$ and so the result eq.(5).
Turbulence is a very active field of research. One knows meanwhile that K41 is not fully correct due to the phenomenon of intermittency. This is similar to the modification of critical exponents in second order phase transitions due to critical fluctuations. [1]

Logarithmic Velocity Profile

In the atmosphere, turbulent flow causes a logarithmic profile for the horizontal velocity $u(z)$ depending upon height z. According to L. Prandtl, we may assume that the typical fluctuations of velocity $\tilde{u}$ are approximately constant, independent of height. $\tilde{u}$ also corresponds to the squareroot of the vertical momentum flow, which we may assume to be constant from one horizontal layer downward to the next. A mixing length L is defined such that the fluctuations $\tilde{u}$ are uncorrelated beyond that length. As the fluid is not moving on the ground, we have a vertical gradient of the average horizontal velocity u, for which we assume the scaling ansatz

$$\frac{du}{dz} = \frac{\tilde{u}}{L} \tag{6}$$

which means that the shearing force $\sim du/dz$ would be constant for constant mixing length. However, as we have a turbulent flow, we will have to allow for bigger eddies with increasing height, simply, as a big role reaches higher up. But since these eddies are responsible for mixing, we may set

$$L = \kappa\ z \tag{7}$$

Inserting this into eq.(6), we obtain

$$\frac{du}{dz} = \frac{\tilde{u}}{\kappa z} \tag{8}$$

which is readily integrated to

$$\frac{u}{\tilde{u}} = \frac{1}{\kappa} log(\frac{z}{z_0}) \tag{9}$$

This is the famous logarithmic velocity profile. z_0 is a position close to the ground (offshore $\approx 1mm$), where due to surface roughness the horizontal velocity is practically zero, $\tilde{u}$ is the typical constant amplitude of velocity fluctuations and $\kappa \approx 0.4$ is the dimensionless von Karman-constant.

Gravitational Surface Waves

Dispersion relation

What is the mimimal requirement for an object to stay afloat on the surface of the sea? Of course the object should be lighter than an equal volume of water, but lighter or heavier can only be discriminated in presence of a gravitational field with gravitational acceleration g! Using g as the only physical parameter and with wavenumber $k = 2\pi/\lambda$ one obtains for the velocity u_p of surface waves (phase-velocity) [1]

$$u_p = \sqrt{\frac{g}{k}} = \frac{\omega}{k} \tag{10}$$

Note that apart from a proportionality factor this is the only way to combine u, k with g in a single equation! The dispersion relation follows with the wave-(angular)-frequency $\omega = u_p\ k$

$$\omega = \sqrt{g\ k} \tag{11}$$

and consequently the group-velocity $u_g = \frac{\partial \omega}{\partial k}$ becomes

$$u_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2} u_p = \frac{\partial \omega}{\partial k} \tag{12}$$

This dispersion is substantially different from acoustic phonons (sound propagation) or photons (light propagation), where the propagation velocity u is constant for all wavelengths (in the limit of long wavelengths or $k \to 0$!)

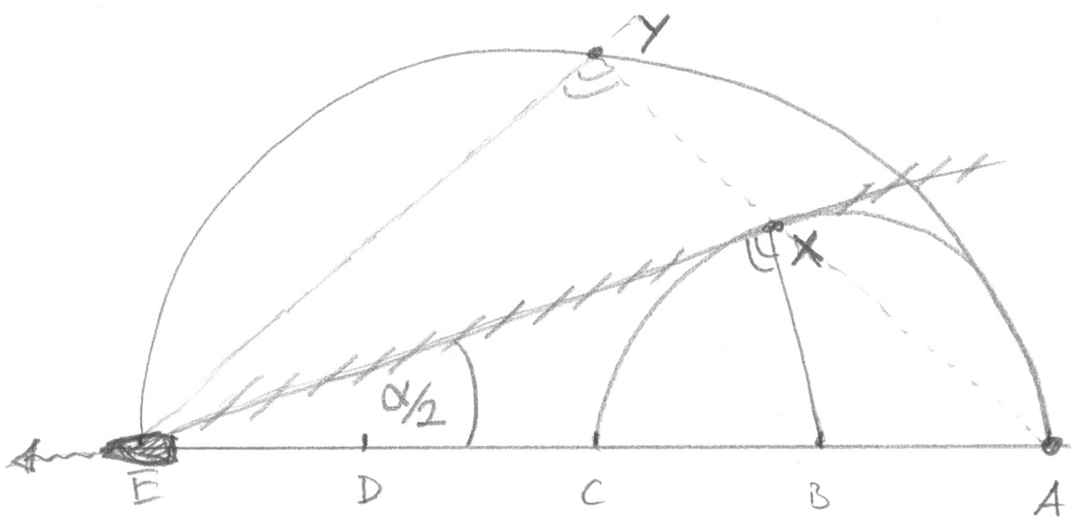
Abbildung 1: Fig. 1a

Kelvin-Angle of Surface Waves

The bow waves of a moving ship are forming a wedge pattern behind the ship, see Fig. 1a. The opening angle α is about 39 degrees. This so-called Kelvin-angle $\alpha = 2\ arcsin(1/3)$ is an immediate consequence of this relation eq.(12) between phase- and group-velocity for gravitational surface waves [3]. The wave-front accordingly is inclined at about $\alpha/2 \approx 19.5$ degrees against the ships axis. Within that wave front, little waves oriented at an angle of 54.7 degrees against the ships axis (moving in direction of 35.3 degrees away from the ships axis) appear behind the front, move through the front (which they form) and disappear before the front.

Explanation, see Fig 1b: In deep water the group velocity of waves is exactly half the phase velocity. Assume that a ship is moving from A to E within time Δt. If a wave-train would be moving from A to Y along the direction A-Y within the same time interval, then the waves would be achored at the bow of the ship in E, the ridge and valleys of the waves being parallel to direction E-Y. (For the moment, point Y could be any point on the semi-circle with center at C, going through A,Y,E). The angle between the orientation of the waves E-Y and their propagation direction A-Y is obviously 90 degrees. Note that the distance A-X is exactly half the distance A-Y, X being on a semi-circle with center at B, going through A and C.

If Y would be close to E, the waves would travel with approximately the speed of the



ship. As the group velocity is only half the phase velocity, there would be no energy transfer away from the ship in this case. Note that the waves propagating at group velocity away from the ship carry energy, hereby producing wave-resistance. Drawing the tangent from E onto the semi-circle of the group velocity, we arrive at the tangent point X, so that A-X corresponds to the group velocity and A-Y to the phase velocity of the wave train.

That tangent E-X is a wavefront and the envelope over the sequence of circular wave-packets being produced by the motion of the ship through the water. From the geometrical construction it is obvious that the angle $\alpha/2$ with the ships axis is obtained from $1/3 = sin(\alpha/2)$, giving $\alpha/2 \approx 19.5$ degrees, and in consequence the angle between the orientation E-Y of the individual waves and the ships axis E-A as 54.7 degrees, as is qualitatively observed in Fig. 1a.

Tsunami

In shallow water with a depth h the dispersion relation (11) must be modified into

$$\omega = \sqrt{g\ k\ tanh(h\ k)} \tag{13}$$

Obviously, the propagation velocities of the waves are reduced. When the depth h is less than about half a typical wavelength, or more precisely $h\ k < 1$, one can linearize the $tanh$ to obtain approximately

$$\omega = k\ \sqrt{g\ h} \tag{14}$$

Note that now the group and phase velocity is identical

$$u_s = \sqrt{g\ h} \tag{15}$$

For an average depth of the oceans of $h \approx 5000$ m, one obtains $u_s \approx 800$ km/h. This is the speed of a Tsunami travelling across an ocean.
Another consequence of the shallow water dispersion is the following. The dispersion for infinite depth (11) allows waves to travel at arbitrary speeds, the wavelength then becoming rather long. For finite depth the modification (15) represents a maximal speed of the surface waves which cannot be exceeded. Wind agitating such a surface at higher speeds, therefore, will generate wedge-patterns like the light cones of Cherenkov-radiation. The random nature of the wind gusts will produce wave-fronts running criss-cross which locally lead to steep waves with discomfort and danger for the ships. [1,4]

Wind against Current

A particular danger between the islands along the north sea coast is encountered, when wind driven ocean waves, coming in eastward from the atlantic ocean, are moving against the westward going ebb-current in the narrow fairways between the islands.
The incoming ocean waves have the square-root dispersion $\omega_o = \sqrt{g\ k_0}$. Between the island those waves have to propagate against the outflowing ebb-current with speed $u_e > 0$. The phase-velocity accordingly will be reduced by that ebb-speed, giving a dispersion relation $\omega_1 = \sqrt{g\ k_1} - u_e\ k_1$, see Fig.2. (On the left of the vertical ω-axis the dispersion of the incoming ocean waves is shown, on the right the dispersion of waves moving against an ebb-current of velocity u_e). Obviously, the frequency must be the same as for the incoming waves $\omega_0 = \omega_1 = \omega$. Then the wavenumber k_1 of the waves travelling against the ebb-current is defined by the relation

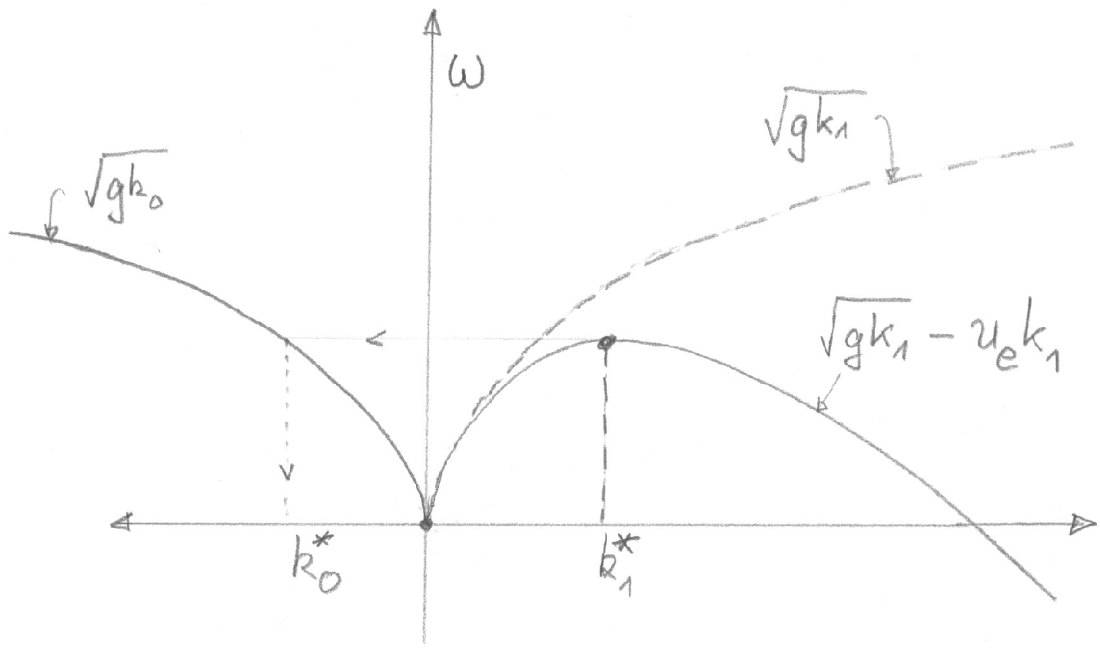
$$\sqrt{g\ k_0} = \sqrt{g\ k_1} - u_e\ k_1 \tag{16}$$

While the left hand side (l.h.s.) of that equation may become arbitrarily large, the right hand side has a maximum at some value k_1^*. At this value, which corresponds to some k_0^*-value for the incoming ocean waves, the derivative of eq.(16) with respect to k_1 and therefore the group velocity becomes zero! (see Fig.2). Since energy is transported by the group velocity this means that the incoming energy at those wavenumbers cannot be transported against the ebb-current and therefore the waves will pile up to enormous amplitudes, a serious danger for shipping. [4]

Shallow-Water Waves

We have encountered shallow-water waves already in the section TSUNAMI. Here now some further details.

Fig. 2.



When the depth h is less than about half a typical wavelength, or more precisely $h\ k < 1$, one can linearize the $tanh$ to obtain approximately

$$\omega = k \sqrt{g\ h} \tag{17}$$

Note that now the group and phase velocity is identical

$$u_s = \sqrt{g\ h} \tag{18}$$

This has several consequences. Waves with different wavelength (but large compared to water-depth!) travel with the same speed, similar to the speed of sound or the speed of light which both do not depend upon the frequency or wavelength. A certain wave pattern then can propagate without changing its shape for some time and distance. A specific feature, so-called solitions, will be shortly discussed below.
In a large sea-area with relatively shallow water like the western part of the Baltic sea, crosswise intersecting seas may appear in strong winds. When the wind speed is appreciably larger than the speed of the shallow water waves, a gust of wind may act on the water surface in analogy of a supersonic jet-plane which leaves a Mach-cone of sound behind. On the water therefore two fronts of waves travel at an angle away from the direction of a localised wind-gust. As there will be many roughly parallel wind-gusts these wave-fronts travel criss-cross in direction oblique to the main wind direction creating an unpleasant surface of cross-waves.
Now to the solitons. In the year 1834 the scottish engineer John Scott Russell was riding on horse-back along an approximately 10 m wide channel. He observed that a single bumpy deformation (about 40 cm high) on the wave surface was travelling a the speed of its horse (about 14 km/h) along the channel maintaining shape and speed

quite constantly over a distance of about 2 miles. In 1895 a mathematical description was given by Korteweg and deVries:

$$0 = \frac{\partial u}{\partial t} + 6 * u * \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \tag{19}$$

A soliton-solution for the amplitude u of this equation is obtained as

$$u(x,t) = (1/2)v * sech^2(\tfrac{\sqrt{v}}{2} * \text{(x - v*t -a)}) \tag{20}$$

with arbitrary constants for the velocity v and the position a, where sech stands for secans hyperbolicus.
The solitions remind us of particles as they can collide with each other and retain their shapes after collisions.

d' Alembert Paradoxon; Kutta-Condition

Why does an airplane wing produce uplift? Its more complicated than described in many textbooks. D'Alembert formulated the following paradox: *Irrotational flow of a non-viscous fluid about an object produces no drag on the object.* And no uplift! This means, you must have viscosity in your medium in order to produce and explain uplift of an airplane wing. Alternatively, one may introduce the Kutta-condition into the flow-equations of an inviscid medium around a wing-structure, see e.g. the NASA-applet FoilsimU [8] (Note: Foilsim does not allow to switch off the Kutta condition, you need FoilsimU).

Asymptotic Matching - Prandtl's Boundary Layer

A flat sheet-metal inside and extending a length L parallel to a viscous flow field produces a perturbed boundary layer of thickness

$$d = (\frac{\nu}{u})^{1/2} \; L^{1/2} \tag{21}$$

where $\nu = \eta/\rho$ is the kinematic viscosity, η the normal viscosity, and u the flow velocity. This is Prandtl's boundary layer.
A powerful theoretical method is the matching of asymptotic expansions [7]. This works in particular for singular perturbations, where a small parameter multiplies the highest derivative, see Nayfeh's book chapter 4.1 [7]

Freak Waves - *Vagues Scelerates*

Freak waves are unusually high waves, which exceed the typical wave height substantially while occuring more frequently than predicted by the conventional wave-theory (linear, Rayleigh, Pierson Moskowitz or JONSWAP-spectrum). The origin and the conditions of generation are not competely resolved. Wave heights of about 30 m have been observed in the oceans. About ten of such waves seem to exist at any instance of time over the world. They seem to arise and to disappear in a relatively short time interval. One model candidate for such freak-waves is based on the nonlinear Schroedinger equation:

$$i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi + \kappa|\psi|^2\psi \qquad (22)$$

for the complex field $\psi(x,t)$. It does not describe the time-evolution of a quantum-state, but for $\kappa < 0$ it allows for so-called bright soliton solutions as well as for breather solutions. Further details are beyond the scope of this article.

Numerical Solution of Navier-Stokes

(see the pages from the manuscript by H. Kopetsch [6])

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7 Ali H. Nayfeh, Perturbation Methods

8 FoilsimU applet: http://www.grc.nasa.gov/ WWW/k-12/aerosim/applet/vj402.html
(select: shape/angle; ideal-flow, then change to no-kutta-condition)

9 See also the program WIND-and-WAVES.html on this web-site.